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# Continuous Mode Tracking Method of Guided Wave in Water-filled Pipe Using Finite Element Analysis

有限要素解析を用いた液体充填パイプの
ガイド波のモード連続追跡法

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## 1. Introduction

The ultrasonic wave propagation in an elongated material might produce the dispersion relation in terms of frequency and wavenumber. The semi-analytical finite element method[1] (SAFE) has been widely used for solving the wave propagation mode with an arbitrary cross-section. In the SAFE, the target region is discretized in the cross-section, while an analytical solution is adopted in the wave propagation direction. The SAFE is formulated in the frequency domain; the dispersion relation can be calculated discretely by sweeping the frequency step by step. Since the dispersion curve becomes tangled for materials that have complicated cross-section and inhomogeneity, it sometimes would be difficult to evaluate the continuity of a mode.

In this study, we propose a mode tracking method that contributes to classifying propagation modes by combining the numerical continuation method[2] (NCM) with the SAFE. Here, algebraic nonlinear equations are formulated from the conventional SAFE. Since the solution set forms a differentiable curve, the NCM tracks the curve continuously and calculates the solutions iteratively.

## 2. Continuous Mode Tracking Method

1.1 Semi-analytical finite element method (SAFE)

We consider the guided wave that propagates in a hollow cylinder. They have many practical applications, including the detection of corrosion in piping. In this study, we assume the cylinder is filled with water. The displacement $\boldsymbol{u}$ in solid satisfies the Navier's equation of motion;

$$\mu_s \nabla^2 \boldsymbol{u} + (\lambda_s + \mu_s) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \boldsymbol{f} = \rho_s \frac{\partial^2 \boldsymbol{u}}{\partial t^2} \quad (1)$$

where $\lambda_s$ and $\mu_s$ are Lame constants, $\rho_s$ is the density, and $\boldsymbol{f}$ is the body force. Also, the pressure $p$ in water can be obtained by solving the following wave equation;

$$\nabla^2 p + f = \frac{\rho_f}{\lambda_f} \frac{\partial^2 p}{\partial t^2} \quad (2)$$

where $\lambda_f$ is the bulk modulus of the fluid and $\rho_f$ is the density. The above equations are solved in cylindrical coordinate $(r, \theta, z)$. Here we adopt an exact analytical solution $e^{in\theta}$ in the circumferential direction. Therefore exact analytical solutions are used in both the $\theta$ and $z$ directions. Here, the displacement and pressure can be represented by[3]

$$\begin{aligned} p(r,\theta,z,t) &= p(r) e^{i(n\theta + kz)} e^{-i\omega t}, \\ \boldsymbol{u}(r,\theta,z,t) &= \boldsymbol{u}(r) e^{i(n\theta + kz)} e^{-i\omega t}, \end{aligned} \quad (2)$$

where $k \in \mathbb{C}$ and $\omega \in \mathbb{R}$ are wavenumber in the wave-propagation direction ( $z$ direction) and circular frequency, respectively.

In the SAFE, the displacement and pressure are discretized into *N*-nodes along the $r$ direction. A system of equations of the conventional SAFE is written in the harmonic wavefield as follows [3]:

$$[\mathbf{K}_1 + ik\mathbf{K}_2 + k^2\mathbf{K}_3 - \omega^2 \mathbf{M}]\mathbf{Q} = 0, \quad (3)$$

where $\mathbf{K}_1, \mathbf{K}_2 \in \mathbb{C}^{N\times N}$, $\mathbf{K}_3, \mathbf{M} \in \mathbb{R}^{N\times N}$, and $\mathbf{Q} \in \mathbb{C}^N$. The relation between $k$ and $\omega$ satisfying Eq. (3) indicates the existence condition of a guided wave. In the conventional method, Eq. (3) is solved as a generalized eigenvalue problem for eigenpair $(k, \mathbf{Q})$ with sweeping $\omega$.

2.2 Numrical contenuous method (NCM)

In this study, we track the solution of Eq. (3) using NCM [2], which can continuously track the solution of a nonlinear equation varying a parameter. However, there are two following problems: $\mathbf{Q}$ corresponds to an eigenvector, and its amplitude and argument of complex are indeterminate. Both real- and complex-valued variables are mixed in Eq. (3). To overcome these problems, we separate all complex-valued variables in Eq. (3) into real and imaginary parts and add constraint conditions for amplitude and argument of the complex of $\mathbf{Q}$. We can write the final form of a system of nonlinear equations as follows:

$$\boldsymbol{h}(\boldsymbol{v}) = \left\{ \begin{matrix} \mathbf{A}\,\Re[\mathbf{Q}] - \mathbf{B}\,\Im[\mathbf{Q}] \\ \mathbf{B}\,\Re[\mathbf{Q}] + \mathbf{A}\,\Im[\mathbf{Q}] \\ |\mathbf{Q}|^2 - 1 \\ \sum_{j=1}^{N} \Im[\mathrm{Q}_j] \end{matrix} \right\} = \mathbf{0}, \quad (4)$$

$$\boldsymbol{v} = \{\Re[\mathbf{Q}], \Im[\mathbf{Q}], \Re[k], \Im[k], \omega\}, \quad (5)$$

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where $\Re[\,]$ and $\Im[\,]$ indicate real and imaginary parts, respectively, and matrices $\mathbf{A}$ and $\mathbf{B}$ are given by

$$\begin{aligned}\mathbf{A} &= \Re[\mathbf{K}_1] - \Re[k]\Im[\mathbf{K}_2] - \Im[k]\Re[\mathbf{K}_2] \\ &\quad + (\Re[k]^2 - \Im[k]^2)\mathbf{K}_3 - \omega^2\mathbf{M}, \\ \mathbf{B} &= \Im[\mathbf{K}_1] + \Re[k]\Re[\mathbf{K}_2] - \Im[k]\Im[\mathbf{K}_2] \\ &\quad + 2\Re[k]\Im[k]\mathbf{K}_3.\end{aligned} \tag{6}$$

Equation (4) are composed of $2N+2$ equations, and $\boldsymbol{v}$ is the $(2N+3)$-dimensional vector. Thus, Eq. (4) and $\boldsymbol{v}$ can be regarded as a curve equation and position vector in $\mathbb{R}^{2N+3}$, respectively. In this study, we use the predictor-corrector algorithm [2] to track $\boldsymbol{v}$ in Eq. (4).

## 3. Numerical Result

We consider an aluminum (longitudinal wave velocity $c_L$: 6300 m/s, transverse wave velocity $c_T$: 3100 m/s, and density: 2750kg/m$^3$) as the target material. The cross-section of the target is an annular shape. The inner and outer diameters of the pipe are $2a$ and $4a$, respectively. The aluminum pipe is filled with water (longitudinal wave velocity: 1480 m/s and density 1000 kg/m$^3$). The displacement and stress in the $r$ direction are continuous at the interface between the aluminum and water. However, the stress in the $\theta$ direction is zero at the interface.

The dispersion relation in the case of $n = 0$ in Eq. (2) is calculated by the continuous mode tracking method. We started with determining each propagation mode's frequency $a\omega / c_T$, which corresponds to the cut-off wavenumber $ak_0$. Here we set $ak_0$ was 0.01. Figure 1(a) and (b) show the frequency – wavenumber curves and group velocity, respectively. A total of 17 modes were obtained in the frequency range $(0 < a\omega / c_T < 10)$. Because of the solid-liquid coupling, the dispersion curves are complicated. As can be seen, all guided wave modes were effectively sorted and separated by the continuous mode tracking method. The decoupling modes labeled as ①, ②, and ③ in Fig.1 are corresponding to SH0, SH1, and SH2 modes, respectively. These decoupling modes can propagate only in the aluminum and do not be affected by the influence of water. On the other hand, the mode labeled as ④ is corresponding to the longitudinal mode and propagates almost only in water.

## 4. Summary

The semi-analytical finite element method (SAFE) is a useful tool to calculate the ultrasonic propagation mode in an elongated material with an arbitrary cross-section. Although the SAFE obtains the dispersion relation of the propagation mode by sweeping the frequency step by step, it was not easy to evaluate the continuity of a mode. This study proposed a mode tracking method that contributes to classifying propagation modes by combining the numerical continuation method (NCM) with the SAFE. The efficiency of the proposed method was verified using a water-filled pipe model. We will show the experimental validation on the conference day.

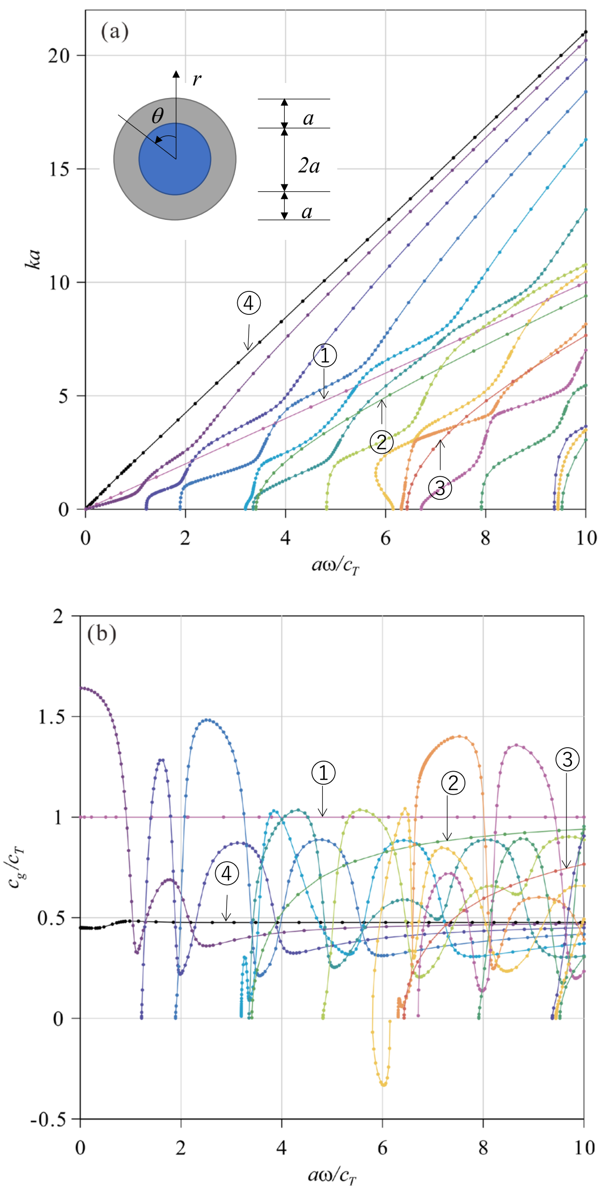


Fig. 1 (a) Frequency – wavenumber curves for an aluminum pipe filled with water. The inner and outer diameters of the pipe are $2a$ and $4a$, respectively. (b) Group velocity dispersion curve.

## References

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