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Differential Forms for the Application to Electromechanical Coupling Systems

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1. Introduction

Instead of the conventional tensor or vector analysis, exterior differential forms (or simply differential forms) can be utilized for the simplification of the mathematical framework: For example, the operations of the gradient, the rotation (curl), and the divergence are unified in a simple manner.

Moreover, the method of differential form enables us to discuss physical quantities more essentially and precisely: For example, although the electric flux density is usually treated as a first-rank tensor or (polar) vector, it is essentially an anti-symmetrical second-rank pseudo (or twisted) tensor[1)].

In this study, the application of the method of differential form to the electromechanical coupling phenomena and transducers is discussed. Some original or improved notations are also introduced for clearer discussion in addition to the conventional notations.

2. Outline of the method of differential form

A differential "n-form" is a kind of inner product between a physical quantity that is expressed with an anti-symmetrical co-variant nth-rank tensor and a contra-variant n-dimensional volume integral element. For example, some typical physical quantities and operators are expressed as differential forms (a bar ($^-$) is added as the notation) as follows :

- Scalar potential $\phi \to \bar{\phi} = \phi \cdot 1 = \phi$ (0-form);
- Electric field $\boldsymbol{E} \to \bar{E} = E_i dx^i = E_1 dx^1 + E_2 dx^2 + E_3 dx^3$ (1-form);
- Nabla (del) operator $\nabla \to \bar{\nabla} = \partial_i dx^i = \partial_1 dx^1 + \partial_2 dx^2 + \partial_3 dx^3$ (1-form), where $\partial_i = \frac{\partial}{\partial x^i}$;
- Electric flux density $\boldsymbol{D} \to \bar{D} = D_{23} dx^2 \wedge dx^3 + D_{31} dx^3 \wedge dx^1 + D_{12} dx^1 \wedge dx^2$ (2-form), where $D_{32} = -D_{23}$, $D_{11} = 0$, etc.;
- Electric charge density or mass density $\rho \to \bar{\rho} = \rho_{123} dx^1 \wedge dx^2 \wedge dx^3$ (3-form) .

In the formulation above, the wedge product denoted by "$\wedge$" is used, which has anti-symmetrical properties such as $dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$, $dx^1 \wedge dx^1 = 0$, $dx^1 \wedge dx^2 \wedge dx^3 = dx^2 \wedge dx^3 \wedge dx^1 = -dx^1 \wedge dx^3 \wedge dx^2$, etc. In general, the wedge product is constructed by anti-symmetrizing the direct product denoted by "$\otimes$" as follows: $\bar{p} \wedge \bar{q} = \bar{p} \otimes \bar{q} - \bar{q} \otimes \bar{p}$, and $\bar{p} \wedge \bar{q} \wedge \bar{r} = \bar{p}\otimes\bar{q}\otimes\bar{r} + \bar{q}\otimes\bar{r}\otimes\bar{p} + \bar{r}\otimes\bar{p}\otimes\bar{q} - \bar{p}\otimes\bar{r}\otimes\bar{q} - \bar{r}\otimes\bar{q}\otimes\bar{p} - \bar{q}\otimes\bar{p}\otimes\bar{r}$.

The star operator or Hodge operator denoted by "$\star$" is used to transform an n-form into a $(3-n)$-form. For example, $\star dx^1 = dx^2 \wedge dx^3$, $\star(dx^3 \wedge dx^1) = dx^2$, $\star(dx^1 \wedge dx^2 \wedge dx^3) = 1$, and the usage of $\star$ twice in a row changes nothing: $\star\star \equiv 1$, and the inverse operator satisfies $(\star)^{-1} \equiv \star$.

The exterior derivative, denoted by "d" usually but "$\bar{\nabla}\wedge$" in this study, provides:

- Operation of the gradient, when $\bar{\nabla}\wedge$ operates on a 0-form, resulting in a 1-form;
- Operation of the rotation (curl), when $\bar{\nabla}\wedge$ operates on a 1-form, resulting in a 2-form;
- Operation of the divergence, when $\bar{\nabla}\wedge$ operates on a 2-form, resulting in a 3-form,

and the usage of $\bar{\nabla}\wedge$ twice in a row results in zero: $(\bar{\nabla}\wedge)(\bar{\nabla}\wedge) \equiv 0$. For example, the exterior derivative of $\bar{\phi}$ (0-form), $\bar{\nabla} \wedge \bar{\phi}$, is calculated as $\partial_i dx^i \wedge (\phi \cdot 1) = \partial_i \phi \, dx^i \wedge 1 = \partial_i \phi \, dx^i$, where $dx^i \wedge 1 = dx^i$ is defined. The result corresponds to the gradient of ϕ expressed with a 1-form. $\bar{\nabla} \wedge (\bar{\nabla} \wedge \bar{\phi})$ is calculated as $\partial_j dx^j \wedge (\partial_i \phi \, dx^i) = (\partial_j \partial_i \phi)(dx^j \wedge dx^i) = 0$, since $\partial_j \partial_i \phi = \partial_i \partial_j \phi$, $dx^j \wedge dx^i = -dx^i \wedge dx^j$, and $\sum_{i,j}$ is conducted.

3. Relations among $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{S}$ (strain), and $\boldsymbol{T}$ (stress)

The notation $\bar{E}_i$ is introduced for describing the i-th term of $\bar{E}$, and similarly the notation $\bar{D}_i$ is also introduced. For example, $\bar{E}_1 = E_1 dx^1$, and $\bar{D}_1 = D_{23}(\star dx^1)$. In the conventional tensor analysis, $\boldsymbol{E}$ and $\boldsymbol{D}$ are both first-rank tensors, and therefore, the operations of $\nabla \cdot \boldsymbol{E}$ and $\nabla \times \boldsymbol{D}$ are admitted.

However, $\bar{E}$ and $\bar{D}$ only accept the operation of rotation and divergence, respectively. The permittivity $\bar{\varepsilon}$ should fulfill a function to transform 1-form into 2-form:

$$\bar{D}_i = \bar{\varepsilon}_i^j \bar{E}_j,$$

$$\bar{\varepsilon}_i^j = \sum_{\text{delayed}} \varepsilon_i^j (\star dx^i)/dx^j,$$

where ε_i^j is an (i,j)-element of the conventional permittivity tensor, and the "delayed sum", denoted by "$\sum_{\text{delayed}}$", is introduced, which requests that the application of Einstein summation convention should be delayed until needed. While $\bar{\nabla} \wedge \bar{E}$ generates $\nabla \times \boldsymbol{E}$, $\bar{\nabla} \wedge \bar{\varepsilon}\bar{E}$ leads to $\nabla \cdot \varepsilon \boldsymbol{E}$.

The strain and stress cannot be represented with simple differential forms. The strain is expressed as

$$\bar{S} = S_{ij} dx^i \otimes dx^j = \frac{1}{2} (\partial_i u_j + \partial_j u_i) dx^i \otimes dx^j,$$

which is the direct product of a 1-form and a 1-form ((1,1)-form). The matrix representation for $\bar{S}$ is

$$\bar{S} = \begin{pmatrix} S_{11}dx^1dx^1 & S_{12}dx^1dx^2 & S_{13}dx^1dx^3 \\ S_{21}dx^2dx^1 & S_{22}dx^2dx^2 & S_{23}dx^2dx^3 \\ S_{31}dx^3dx^1 & S_{32}dx^3dx^2 & S_{33}dx^3dx^3 \end{pmatrix},$$

which is a symmetrical matrix.

In the definition of the stress, since the non-rotational force T_{ij} in the j-direction on the i-plane (per unit area) is considered, the stress is expressed as

$$\bar{T} = T_{ij} (\star dx^i) \otimes dx^j = T_{ji} (\star dx^j) \otimes dx^i,$$

which is the direct product of a 2-form and a 1-form ((2,1)-form). The stiffness $\bar{c}$ that transforms $\bar{S}$ in (1, 1)-form into the (2, 1)-form should have the following form:

$$\bar{T}_{ij} = \bar{c}_{ij}^{kl} \bar{S}_{kl},$$

$$\bar{c}_{ij}^{kl} = \sum_{\text{delayed}} c_{ij}^{kl} \frac{(\star dx^i) \otimes dx^j}{dx^k \otimes dx^l}$$

where c_{ij}^{kl} is the conventional stiffness tensor.

By taking an exterior derivative on (the part of the 2-form of) $\bar{T}$,

$$\begin{aligned} \bar{\nabla} \wedge \bar{T} &= \partial_k T_{ij} (dx^k \wedge (\star dx^i)) \otimes dx^j \\ &= \partial_k T_{ij} \delta^{ik} dV \otimes dx^j, \end{aligned} \tag{1}$$

is obtained ((3,1)-form), where only the case of $i = k$ remains in $dx^k \wedge (\star dx^i)$, which forms $dV \equiv dx^1 \wedge dx^2 \wedge dx^3$ (3-form), and δ^{ik} is the Kronecker delta.

The calculation of $\bar{\nabla} \wedge \bar{c}\bar{S}$ can be performed similarly, by considering that $\bar{c}\bar{S}$ is now a (2,1)-form.

According to Newton's second law, eq. (1) is equivalent to

$$\bar{\rho} \otimes {\partial_t}^2 \bar{u} = \rho \, {\partial_t}^2 u_j dV \otimes dx^j \quad \text{((3,1)-form)},$$

where ${\partial_t}^2 = \frac{\partial^2}{\partial t^2}$, $\bar{\rho} = \rho \, dV$ is the differential form of the mass density (3-form), and $\bar{u} = u_i dx^i$ is the differential form of the displacement (1-form).

The electromechanical coupling phenomena can be described as

$$\bar{D}_i = \bar{d}_i^{jk} \bar{T}_{jk} \qquad \text{(direct effect)},$$

$$\bar{S}_{jk} = \bar{d'}_{jk}^{i} \bar{E}_i \quad \text{(converse effect)},$$

where

$$\bar{d}_i^{jk} = \sum_{\text{delayed}} d_i^{jk} \frac{(\star dx^i)}{(\star dx^j) \otimes dx^k},$$

$$\bar{d'}_{jk}^{i} = \sum_{\text{delayed}} d_{jk}^{i} \frac{dx^j \otimes dx^k}{dx^i}.$$

The conventional d-constants (the subscripts and superscripts are omitted here) for the direct and converse effects are identical in value and dimension reciprocally, since $d = -\partial^2 G/(\partial T_{jk} \partial E_i)$ with $D_i = \partial G/\partial E_i$ and $S_{jk} = \partial G/\partial T_{jk}$ for Gibbs free energy G; however, we find that $\bar{d}_i^{jk} \neq \bar{d'}_{jk}^{i}$ from the viewpoint of differential form.

The relations among $\bar{E}$, $\bar{D}$, $\bar{S}$ and $\bar{T}$ in an electromechanical transducer can be described, in general, by using a function f as

$$f(\bar{E}, \bar{D}, \bar{S}, \bar{T}) = 0 \rightarrow \bar{\nabla} \wedge f(\bar{E}, \bar{D}, \bar{S}, \bar{T}) = 0, \tag{2}$$

in which the boundary conditions are reflected by considering which elements of $\bar{E}$, $\bar{D}$, $\bar{S}$, $\bar{T}$, and $\bar{\nabla}$ remain or disappear. The wave equation with regard to $\bar{u}$ can be derived from eq. (2), using $\bar{\nabla} \wedge \bar{T} = \bar{\rho} \otimes {\partial_t}^2 \bar{u}$, $\bar{\nabla} \wedge \bar{E} = 0$ ($\nabla \times \boldsymbol{E} = \boldsymbol{0}$), and $\bar{\nabla} \wedge \bar{D} = 0$ ($\nabla \cdot \boldsymbol{D} = 0$), and then the acoustic velocity can be obtained. Different operations of derivative are unified by using the exterior derivative, which is one of the advantages of the method of differential form.

References

1. F. W. Hehl and Y. N. Obukhov: Foundations of Classical Electrodynamics, Charge, Flux, and Metric (Birkhäuser, 2003).